Algorithmic and methodological developments towards full waveform inversion in 3D elastic media

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SUMMARY

We present some methodological aspects of full waveform inversion (FWI) in 3D elastic media. FWI is a local optimization scheme aiming to update the initial model with perturbations in order to minimize the misfit between the computed and observed data. The pertubations are found in the opposite direction of the gradient which can be evaluated efficiently with the adjoint state method. While this method is generally applied to a second order expression of the wave equation, we develop our inversion scheme on a first-order velocity-stress formulation which provides a great flexibility to recast the wave equation in pseudo-conservative form. We show how to take advantage of this formalism to develop the gradient of the misfit function with the adjoint-state method in a straightforward way. The inversion is implemented in the frequency domain, while the seismic modeling is performed in time. We propose also an abstraction concept between the forward and inverse problems that allows us to use different modeling engines in the inversion code and to perform target-oriented imaging.

INTRODUCTION

Full waveform inversion (FWI) is one of the most promising techniques for seismic imaging. It relies on a formalism that allows to take into account the full information content of the data (Tarantola, 1984) as opposed to more classical techniques such as travel time tomography. As a result, FWI is a high resolution imaging process able to reach a spatial accuracy of half a wavelength (Sirgue and Pratt, 2004). FWI is based on a local optimization scheme, where the gradient of the misfit function can be computed efficiently with the adjoint-state method (Plessix, 2006). FWI is an ill-posed problem, that requires the starting model to be close enough to the real one in order to converge to the global minimum. Another counterpart of FWI is the required computational ressources when considering models and frequencies of interest. The task becomes even more challenging when one attempts to perform the inversion using the elastic equation (Shi et al., 2007; Brossier et al., 2009) instead of using the acoustic approximation (Mulder and Plessix, 2008; Barnes and Charara, 2009). This is the reason why until recently most studies were limited to 2D cases (e.g., Ravaut et al., 2004). In the last few years, due to the increase of the available computational power, FWI has focused a lot of interests and continuous efforts towards inversion of 3D data sets. Remarkable applications have been done in 3D using the acoustic approximation (Plessix, 2009; Sirgue et al., 2010; Plessix and Perkins, 2010) but the extension to the 3D elastic case is still an ongoing work. In this study, we present two innovative features specifically developped for 3D elastic media. The first one refers to the concept of abstraction between the forward and inverse problems in order to adapt easily the numerical modeling method to the characteristics of the medium. The second feature deals with a pseudo-conservative formalism which makes the implementation of the adjoint state method straightforward.

In the remainder of this study, we introduce the wave equations in the pseudo-conservative form and we develop the expression of the gradient of the misfit function using the adjoint state method in the time domain. Finally, we discuss the algorithmic implementation of the method and we conclude with a validation of our code with an application to a target of the 3D SEG/EAGE Overthurst model.

FORWARD PROBLEM EQUATIONS

We consider the 3D velocity-stress elastodynamic wave equations without attenuation for seismic wave modeling where the sources are either punctual forces $(s_{f_x}, s_{f_y}, s_{f_z})$ or external stresses $(s_{\sigma_{xx}}, s_{\sigma_{yy}}, s_{\sigma_{zz}}, s_{\sigma_{xy}}, s_{\sigma_{xz}}, s_{\sigma_{yz}})$ applied to elementary surfaces. These excitations are denoted by the vector

$$\mathbf{s} = \left(s_{f_x}, s_{f_y}, s_{f_z}, s_{\sigma_{xx}}, s_{\sigma_{yy}}, s_{\sigma_{zz}}, s_{\sigma_{xy}}, s_{\sigma_{xz}}, s_{\sigma_{yz}}\right)^T.$$

These fields satisfy the elastodynamic equations which can be recast in compact form as

$$\partial_t \mathbf{u} = \mathbf{A}\mathbf{u} + \mathbf{s}, \tag{1}$$

where the vector $\mathbf{u}$ is given by

$$\mathbf{u} = \left(v_x, v_y, v_z, \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\right)^T. \tag{2}$$

In isotropic media, one can apply a change of variables through a linear operator $\mathbf{T}$ to write the system in conservative form, where all the medium properties are on the left side of equations.

$$\Lambda \partial_t \mathbf{w} = \mathbf{A}'\mathbf{w} + \mathbf{s}', \tag{3}$$

where Λ is a diagonal matrix given by

$$diag(\Lambda) = \left(\rho, \rho, \rho, \frac{1}{3\lambda + 2\mu}, \frac{1}{2\mu}, \frac{1}{2\mu}, \frac{1}{2\mu}, \frac{1}{2\mu}, \frac{1}{2\mu}\right). \tag{4}$$

The new components of the wavefield $\mathbf{w} = \mathbf{T}\,\mathbf{u}$ are given by

$$\begin{aligned} \mathbf{w} \quad = \quad & (v_x, v_y, v_z, \frac{1}{\sqrt{3}}\mathrm{Tr}(\sigma), \frac{\sqrt{3}}{\sqrt{2}}(\sigma_{zz} - \frac{1}{3}\mathrm{Tr}(\sigma)), \\ & \frac{1}{\sqrt{2}}(-\sigma_{xx} + \sigma_{yy}), \sqrt{2}\sigma_{xy}, \sqrt{2}\sigma_{xz}, \sqrt{2}\sigma_{yz})^t, \end{aligned} \tag{5}$$

where the velocity components are left unchanged, while the stress tensor has been modified. Matrices are related by the relation

$$\mathbf{A}' = \Lambda \mathbf{T}\mathbf{A}\mathbf{T}^{-1}. \tag{6}$$

The new matrix is symmetric and does not depend on the physical properties of the medium. The corresponding source term is

$$\mathbf{s}' = \Lambda \mathbf{T}\mathbf{s}. \tag{7}$$

THE ADJOINT STATE METHOD

In a Hilbert space, one defines a scalar product related to a norm such that, for two vectors of this Hilbert space namely f and g, one can write

$$\langle f \mid g \rangle = \int_{\Omega} f^*(x) g(x) dx, \tag{8}$$

where the operation denoted as * is the transpose of the conjugate of f. We consider a bounded space Ω. The norm is such that, if $\langle f \mid f \rangle = 0$, f is the null element of the space.

Adjoint Operators

Let us consider a linear operator M applied to the vector g. The adjoint operator $M^\dagger$ of M is defined as,

$$\langle f, Mg \rangle = \langle M^\dagger f, g \rangle. \tag{9}$$

Auto-adjoint operators are defined by $M^\dagger = M$. For finite discrete space, the adjoint operation is equivalent to the transpose of the conjugate of the matrix M.

Adjoint State Equation for the Conservative Formulation

The adjoint state method (Lions, 1972; Chavent, 2009) is a well known technique in inverse problem theory with many geophysical applications (Plessix, 2006). Even though setting up the adjoint state problem requires additional work to solve for an adjoint variable which has no primary physical interest in the solution of the inverse problem, this method is appealing because the computation of the gradient with respect to a model parameter requires two evaluations of the partial differential equations. The alternative method, which consists in the explicit computation of the Fréchet derivatives, is expensive to compute, as it requires one forward modeling for each non redundant position of source and receiver (Shin et al., 2001).

Our aim is to develop the gradient of the misfit function from the pseudo-conservative form of the wave equation to simplify the numerical implementation of the gradient. Indeed, we have shown that, when the pseudo-conservative form is used, the forward modeling operator is self-adjoint. Moreover, the radiation pattern matrix in the kernel of the gradient (Pratt et al., 1998) is diagonal and, therefore easy to implement in a parallel environment. Computing the gradient of the misfit function from the pseudo-conservative form of the wave equation requires the solution of the state equation and of the adjoint-state equation written in the pseudo-conservative form. On the other hand, we wish to perform seismic modeling using the non-conservative form of the velocity-stress wave equation, which is suitable for conventional modeling schemes. We shall show how to infer the solution of the pseudo-conservative state equations and of the adjoint-state equations from the numerical solutions of the non-conservative wave equation. This amounts to adapting the source term of the non-conservative wave equation.

Let us define the misfit function

$$J(\mathbf{u}, \mathbf{m}) = \frac{1}{2} \langle \mathbf{Ru} - \mathbf{d}_{obs} | \mathbf{Ru} - \mathbf{d}_{obs} \rangle_Y, \tag{10}$$

where $u(x,t) \in W$, $W = \{f : \mathbf{R}^3 \times [0 \times T] \longrightarrow \mathbf{R}^3,\ s.t\ u(x,0) = 0\}$, and $\mathbf{d}_{obs}$ are the observed data recorded at receiver positions. In equation (10), the subspace Y of W spans over time and receiver positions. We have assumed only one source, but this can be easily generalized by summation over source positions. The operator R is a restriction operator on the receiver positions for the components we consider in the misfit function. In the framework of local optimization, minimization of J with respect to model parameters $\mathbf{m} = (m_i) \in M$ in the vicinity of a starting model $\mathbf{m}_0$ requires to compute the gradient of J for $\mathbf{m} = \mathbf{m}_0$. In this study, we compute the gradient of J with the adjoint state method.

Let's introduce the augmented functional $L : W \times W \times W^* \times W^* \times M \to R$ with equality constraints (3) and $\mathbf{w} = \mathbf{Tu}$,

$$\begin{aligned} L(\mathbf{u}, \mathbf{w}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{m}) &= J(\mathbf{u}) + \left\langle \mathbf{p}_1 \mid \left(\Lambda \partial_t \mathbf{w} - \mathbf{A}'\mathbf{w} - \mathbf{s}' \right) \right\rangle_W \\ &+ \langle \mathbf{p}_2 \mid \mathbf{w} - \mathbf{Tu} \rangle_W, \end{aligned} \tag{11}$$

where $\langle f(x,t) g(x,t) \rangle_W = \int_\Omega \int_0^T f^*(x,t) g(x,t) dt dx$ and fields u, w, p_1, p_2 and m, are assumed independent. The dependence between them will be found later at the minimum. The fields $p_i(x,t)$ are the adjoint fields belonging to the dual space of W, *i.e.* $w, u \in W$ and $p_1, p_2 \in W^*$. The adjoint fields p_i can also be referred to as Lagrange multipliers associated with the Lagrangian L. The elements of W that satisfy the state (wave) equation are called realizations of the state equation. At the saddle points (u, p_i) of the Lagrangian L, $\frac{\partial L}{\partial p_i} = 0$ and $\frac{\partial L}{\partial \mathbf{u}} = 0$, the state and the so-called adjoint state equations are satisfied, which leads to

$$\frac{\partial J}{\partial \mathbf{m}} = \frac{\partial L}{\partial \mathbf{m}}. \tag{12}$$

Equation 12 provides an explicit expression of the gradient of J as a function of the state and adjoint state variables.

We now evaluate each of these expressions.

State equations

Zeroing the derivative of the Lagrangian with respect to the Lagrange multipliers leads to the state equation:

$$\begin{aligned} \partial L / \partial \mathbf{p}_1 &= 0 \\ \Lambda \partial_t \mathbf{w}(x,t) - \mathbf{A}'\mathbf{w}(x,t) &= \mathbf{s}'(t), \end{aligned} \tag{13}$$

which is exactly the forward problem equation in the conservative form (equation 3). The other restriction is found by

$$\begin{aligned} \partial L / \partial \mathbf{p}_2 &= 0 \\ \mathbf{w} &= \mathbf{T}\,\mathbf{u}, \end{aligned} \tag{14}$$

that corresponds to the change of variable required to transform the wave equation in pseudo-conservative form.

Adjoint state equations

The relation $\partial L / \partial w = 0$ requires a rewriting of the Lagrangian using derivation by parts which gives the relation

$$\begin{aligned} L(\mathbf{u}, \mathbf{w}, \mathbf{p}_1, \mathbf{p}_2, m) &= J(\mathbf{u}) + \int_\Omega \left(\mathbf{p}_1(T) \Lambda \mathbf{w}(T) - \mathbf{p}_1(0) \Lambda \mathbf{w}_0 \right) dx \\ &- \langle \partial_t \mathbf{p}_1 | \Lambda \mathbf{w} \rangle_W - \left\langle \mathbf{p}_1 \middle| \left(\mathbf{A}'\mathbf{w} + \mathbf{s}'(t) \right) \right\rangle_W + \langle \mathbf{p}_2 \mid \mathbf{w} - \mathbf{Tu} \rangle_W, \end{aligned} \tag{15}$$

where we have used the initial condition $\mathbf{w}(0) = \mathbf{w}_0$. We can find the derivative if we perturb the variable $\mathbf{w}$ in the Lagrangian in an arbitrary direction $\mathbf{z}$, giving us the following definition

$$\frac{\partial L}{\partial \mathbf{w}} \cdot \mathbf{z} = \lim_{\varepsilon \to 0} \frac{L(\mathbf{w} + \varepsilon \mathbf{z}) - L(\mathbf{w})}{\varepsilon}, \tag{16}$$

leading to the expression

$$\begin{aligned} L(\mathbf{w} + \varepsilon \mathbf{z}) - L(\mathbf{w}) &= \varepsilon \mathbf{p}(T) \Lambda \mathbf{z}(T) - \varepsilon \langle \partial_t \mathbf{p}_1 | \Lambda\, \mathbf{z} \rangle_W \\ &- \varepsilon \left\langle \mathbf{p}_1 | \mathbf{A}'\, \mathbf{z} \right\rangle_W + \varepsilon \langle \mathbf{p}_2 |\, \mathbf{z} \rangle_W . \end{aligned} \tag{17}$$

Following the definition (16), one can get the expression

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}} \cdot \mathbf{z} &= -\left\langle \Lambda^\dagger \partial_t \mathbf{p}_1 | \mathbf{z} \right\rangle_W - \left\langle \mathbf{A}'^\dagger \mathbf{p}_1 | \mathbf{z} \right\rangle_W \\ &+ \int_\Omega \mathbf{p}_1(T) \mathbf{z}(T) dx + \langle \mathbf{p}_2 |\, \mathbf{z} \rangle_W . \end{aligned} \tag{18}$$

If we impose that this derivative must be zero, whatever the value of $\mathbf{z}$, one must impose the final condition $\mathbf{p}_1(T) = 0$, from which we infer the first adjoint state equation for $\mathbf{p}_1$,

$$\Lambda^\dagger \partial_t \mathbf{p}_1 + \mathbf{A}'^\dagger \mathbf{p}_1 = \mathbf{p}_2 . \tag{19}$$

Since the diagonal matrix Λ is symmetric and real and $\mathbf{A}'^\dagger = -\mathbf{A}'$, the equation (19) can be simplified into

$$\Lambda \partial_t \mathbf{p}_1 - \mathbf{A}' \mathbf{p}_1 = \mathbf{p}_2 . \tag{20}$$

The first-adjoint state equation shows that the field $\mathbf{p}_1$ satisfies the conservative wave equation, where the source term is the adjoint-state variable $\mathbf{p}_2$. From the third condition $\partial L / \partial \mathbf{u} = 0$, we find the relation

$$\frac{\partial L}{\partial \mathbf{u}} \cdot \mathbf{z} = \frac{\partial J}{\partial \mathbf{u}} \cdot \mathbf{z} - \langle \mathbf{p}_2 | \mathbf{T} \mathbf{z} \rangle_W . \tag{21}$$

Using the definition of the misfit function one can write,

$$\frac{\partial J}{\partial \mathbf{u}}(\mathbf{u}, \mathbf{m}) = \mathbf{R}^\dagger (\mathbf{R}\mathbf{u} - \mathbf{d}_{obs}). \tag{22}$$

By using the minimality condition in (21):

$$\mathbf{R}^T (\mathbf{R}\mathbf{u} - \mathbf{d}_{obs}) = \mathbf{T}^T \mathbf{p}_2 . \tag{23}$$

Using this value of the field $\mathbf{p}_2$ in the equation (20), we obtain the expression:

$$\Lambda \partial_t \mathbf{p}_1 - \mathbf{A}' \mathbf{p}_1 = \mathbf{T}^{T^{-1}} \mathbf{R}^T (\mathbf{R}\mathbf{u} - \mathbf{d}_{obs}), \tag{24}$$

which are the partial differential equations of the adjoint field $\mathbf{p}_1$. Using the relation (6), we transform back this equation into a non-conservative form as

$$\partial_t \mathbf{q}_1 - \mathbf{A} \mathbf{q}_1 = (\mathbf{T}^T \Lambda \mathbf{T})^{-1} \mathbf{R}^T (\mathbf{R}\mathbf{u} - \mathbf{d}_{obs}), \tag{25}$$

with the introduction of the new field $\mathbf{q}_1$ defined by the relation $\mathbf{p}_1 = \mathbf{T}\,\mathbf{q}_1$. This equation allows to compute this new adjoint wavefield for the non-conservative system with a specific source term. Since we imposed $\mathbf{q}_1(T) = 0$, we have to solve the problem from a final time to the initial time considering the RHS has the excitation.

Gradient of the misfit function

The last condition deals with the derivative of the Lagrangian with respect to model parameters $\mathbf{m}$. Let us first rewrite the Lagrangian as

$$\begin{aligned} L(\mathbf{u}, \mathbf{w}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{m}) &= J(\mathbf{u}) + \left\langle \mathbf{p}_1 \mid \left(\Lambda (\partial_t \mathbf{w} - \mathbf{T}\mathbf{s}) - \mathbf{A}'\mathbf{w} \right) \right\rangle_W \\ &+ \langle \mathbf{p}_2 \mid \mathbf{w} - \mathbf{T}\mathbf{u} \rangle_W , \end{aligned} \tag{26}$$

where we have used the relation (7) to write explicitly the dependence of the source on the parameters. We perturb the model in a certain direction following the same procedure as before. We end up with the relation

$$L(\mathbf{m} + \varepsilon \mathbf{z}) - L(\mathbf{m}) = \varepsilon \left\langle \mathbf{p}_1(t) | \frac{\partial \Lambda}{\partial m_i} \mathbf{z} (\partial_t \mathbf{w} - \mathbf{T}\mathbf{s}) \right\rangle_W \tag{27}$$

Dividing by the small parameter gives the expression of the gradient $\partial L / \partial m_i$. Using condition (12) we infer

$$\frac{\partial J}{\partial m_i}(\mathbf{u}) = -(\partial_t \mathbf{w} - \mathbf{T}\mathbf{s})^T \left(\frac{\partial \Lambda}{\partial m_i} \right)^T \mathbf{p}_1 , \tag{28}$$

where the summation is performed on W, namely, over time and space. Eq. (28) can be expressed in the final non-conservative expression

$$\frac{\partial J}{\partial m_i}(\mathbf{u}) = -(\partial_t \mathbf{u} - \mathbf{s})^T \mathbf{T}^T \left(\frac{\partial \Lambda}{\partial m_i} \right)^T \mathbf{T}\mathbf{q}_1 . \tag{29}$$

This gradient expression is the one we could implement numerically when considering a time domain formulation. Of note, in equation (29), $\partial \Lambda / \partial \mu_i$ is infinite and the corresponding incident and adjoint (shear) stress fields are zero in liquid ($\mu = 0$). From a theoretical point of view, the ability to update the shear velocity from an initial liquid model in our FWI formulation remains an open question. However, this is of limited practical interest, since the water layer is known in marine environment, and is generally kept fixed during FWI.

The gradient will be found using the same procedure in the frequency domain. We directly jump into the final expression of the gradient as

$$\frac{\partial J}{\partial m_i}(\mathbf{u}) = \Re \left(\sum_\omega (i\omega \mathbf{u} + \mathbf{s})^\dagger \mathbf{T}^T \left(\frac{\partial \Lambda}{\partial m_i} \right)^\dagger \mathbf{T}\mathbf{q}_1 \right), \tag{30}$$

which is the gradient we estimate in the inversion in the frequency domain. Let us remark that (a) the influence of the source term is concentrated at the source and that (b) the gradient is a pure local expression easing the numerical implementation. For the acoustic case, we proceed with the restriction on the pressure for which we shall provide an application in the next section.

Algorithm

In summary, the workflow will be as following:

1. Compute the incident wavefield $\mathbf{u}$ with the equation (1).
2. Compute the data residuals: $\mathbf{R}(\mathbf{u}) - \mathbf{d}_{obs}$.
3. Compute the adjoint wavefield $\mathbf{q}_1$ from the non-conservative wave equation with the equation (25).
4. Compute the gradient of the misfit function with the equation (30).

ABSTRACTION FOR DIRECT & INVERSE PROBLEMS

The discretizations of the forward and inverse problems can be different. Our abstraction concept is based on this principle and mainly consists in forward and backward projections between the modeling and inverse discrete models. The abstraction concept in a parallel environment is depicted in the figure 1 which represents a schematic view in 2D. At each iteration, the incident and back-propagated wavefields are computed via equations (3) and (25). The frequency solutions are extracted with a discrete Fourier transform during the time steps as initially proposed by (Sirgue et al., 2008). Once the modeling is over, the frequency wavefields are projected onto the inverse discretization (arrow 1 in Fig. 1). The computation of the gradient is done through the expression (30), which is totally independent from the numerical method used in the forward modeling. When the new model has been evaluated, the physical properties are then projected onto the forward modeling discretization (arrow 2 in Fig. 1). After this last projection, the search of the new model can be done with the classical sequence of forward modelings in order to retrieve the model that minimizes the objective function.

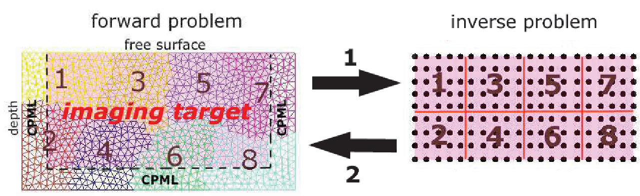

Figure 1: Illustration of the abstraction principle between the forward problem (left) and inverse problem (right).

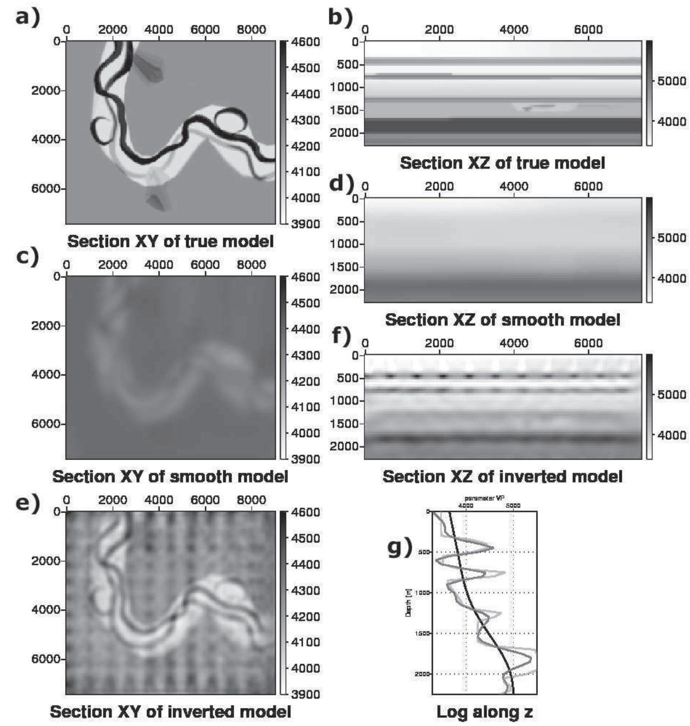

Figure 2: (a-b) Horizontal (a) and vertical (b) sections of the SEG/EAGE Overthrust model; (c-d) Same as (a-b) for the initial model; (e-f) Same as (a-b) for the FWI model. g) Log along z axis in the middle of the model - The black, light gray, and drak gray curves are from the initial model, the true model and the FWI model, respectively. Note the footprint of the coarse acquisition in the FWI model.

INVERSION IN THE SEG/EAGE OVERTHRUST MODEL

We have applied our FWI algorithm to a limited target of the SEG/EAGE overthrust model in the acoustic approximation. Seismic modeling is performed with an acoustic velocity-stress $\mathscr{O}(\Delta t^2, \Delta x^4)$ finite-difference time-domain staggered grid method. Some preliminary results (Fig. 2) are obtained for a coarse acquisition involving 11×10 sources and 75×91 receivers located at the top of the model. The spacing between sources is 700 m in the x- and y-directions, respectively. The spacing between receivers is 100 m in both directions. We used two groups of frequencies: the first one with 8 frequencies ranging from 3 to 8 Hz and the second with 8 frequencies ranging from 6.5 to 12.5 Hz. Twelve iterations have been performed for each frequency group. The initial model has been obtained with a Gaussian smoothing of the true model. The final results exhibit the main features of the model, especially the channel structure, although the footprint of the coarse acquisition is clearly visible in the FWI model. Comparison between vertical profiles of the true and of the FWI models shows a good quantitative reconstruction of the velocities. The computation time is about 24 h with 128 CPUs.

CONCLUSIONS AND PERSPECTIVES

We have presented methodological developments specific to 3D FWI in elastic media using standard equations as well as conservative equations. We have shown how to estimate the gradient through the adjoint method and we have illustrated the interest of the conservative approach avoiding model parameter spatial derivatives whatever are values of medium parameters. This gradient has been applied to a toy synthetic example in order to illustrate its efficiency. We are currently applying the proposed FWI tool to more realistic targets.

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Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2011 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

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